

Gyrating Schrödinger Geometries and Non-Relativistic Field Theories

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ABSTRACT

We propose homogeneous metrics of Petrov type III that describe gyrating Schrödinger geometries as duals to some non-relativistic field theories, in which the Schrödinger symmetry is broken further so that the phase space has a linear dependence of the momentum in a selected direction. We show that such solutions can arise in four-dimensional Einstein-Weyl supergravity as well as higher-dimensional extended gravities with quadratic curvature terms coupled to a massive vector. In Einstein-Weyl supergravity, the gyrating Schrödinger solutions can be supersymmetric, preserving $\frac{1}{4}$ of the supersymmetry. We obtain the exact Green function in the phase space associated with a bulk free massive scalar.

1 Introduction

The AdS/CFT correspondence provides a way of relating d -dimensional relativistic conformal field theories in the strongly-coupled regime to bulk theories including gravity in $d + 1$ dimensions. It is also of considerable interest, in view of the possible applications in condensed matter and other branches of physics, to study situations where the boundary of the gravitational theory does not possess the full relativistic symmetries of a d -dimensional CFT. For this reason, $(d + 1)$ -dimensional gravitational backgrounds that are asymptotic not to anti-de Sitter spacetime, but rather to so-called or Schrödinger [1, 2] or Lifshitz spacetimes [3], have been considered. In the Lifshitz case the boundary spacetime exhibits a scaling behaviour in the time direction, characterised by the Lifshitz exponent z , that differs from the $z = 1$ scaling in the spatial directions. Thus when $z \neq 1$ there is an anisotropy between the time direction and the spatial directions. In the Schrödinger case the coordinate identified as time in the boundary theory enters via only a first derivative when one considers wave equations in the bulk background, thus leading to the expectation that boundary theory will represent a quantum-mechanical system described by the non-relativistic Schrödinger equation.

A key ingredient in interpreting the nature of the boundary theory is to study the symmetry group of the bulk system. In the case of a $(d + 1)$ -dimensional AdS bulk solution, this symmetry will be the full conformal group $SO(d, 2)$. The asymptotic symmetry of a Lifshitz or Schrödinger type spacetime will be reduced to some subgroup of $SO(d, 2)$.

In this paper, we shall consider situations where the symmetry group exhibits not only an anisotropy between time and the spatial directions, but in addition an anisotropy within the spatial directions themselves. Such an anisotropy could arise, for example, if there were a uniform electric or magnetic field along some particular direction in the dual boundary system. Our motivation for investigating systems exhibiting a spatial anisotropy arose from our finding that the associated bulk gravitational backgrounds turn out to arise naturally in theories of gravity where the usual Einstein-Hilbert action is augmented by higher-order curvature contributions. The metrics we consider are of dimension $D = 4 + n$, and take the form

$$ds^2 = \ell^2 \left[\frac{dr^2 - 2dudv + dx^2 + dy^i dy^i}{r^2} + \frac{2c_1 dudx}{r^{z+1}} - \frac{c_2 du^2}{r^{2z}} \right], \quad (1)$$

where $1 \leq i \leq n$ and c_i are constants. They have 1) shift, 2) rotation, 3) boost and 4) dilatation symmetries given by

$$1) : \quad \delta u = a^+, \quad \delta v = a^-, \quad \delta x = a_x, \quad \delta y_i = a_i;$$

$$\begin{aligned}
2) : \quad & \delta y_i = \Lambda_{ij} y_j, \quad \Lambda_{ij} = \Lambda_{[ij]}; \quad 3) : \quad \delta y_i = b_i u, \quad \delta v = b_i y_i; \\
4) : \quad & \delta x = wx, \quad \delta y_i = wy_i, \quad \delta r = wr, \quad \delta u = zu, \quad \delta v = (2 - z)wv. \quad (2)
\end{aligned}$$

The metrics are homogeneous, and all curvature invariants are unchanged from their values in the AdS metrics that have $c_1 = c_2 = 0$. If $c_1 = 0$, the metrics reduce to standard Schrödinger metrics. In general c_i can be functions of all coordinates except the null coordinate v . The corresponding metrics describe gyrating spacetimes. AdS gyratons were studied in [4].

We have investigated in some detail the case of four-dimensional gravity with a quadratic curvature addition proportional to the square of the Weyl tensor. This theory, known as Einstein-Weyl gravity, admits a supersymmetric extension to an off-shell theory of supergravity. We find that for suitable choices of parameters our solutions are supersymmetric. The four-dimensional metrics are in general of Petrov type III, degenerating to Petrov type N if $c_1 = 0$ or $z = 1$ or 0 .

We also find solutions with spatial anisotropy in arbitrary higher dimensions. We have studied two different higher-derivative theories in higher dimensions, namely Einstein gravity augmented by the four-dimensional Gauss-Bonnet invariant, and Einstein gravity instead augmented by the addition of the Weyl-squared invariant.

Having obtained the gravitational backgrounds, we then consider a minimally-coupled probe scalar field and compute the associated 2-point functions in the boundary field theories.

2 Einstein-Weyl Supergravity and BPS solutions

The field content of the off-shell $\mathcal{N} = 1$, $D = 4$ supergravity consists of the vielbein e_μ^a , a vector A_μ and a complex scalar $S + iP$, totalling 12 off-shell degrees of freedom, matching with that of the off-shell gravitino ψ_μ . If one just considers the supersymmetric extension of ordinary Einstein gravity, then the fields A_μ , S and P are auxiliary with purely algebraic equations of motion [5, 6]. In the supersymmetric extension of Einstein-Weyl gravity [7, 8], the field A_μ becomes a dynamical massive vector, while S and P are still auxiliary. Adopting the notation of [8], the bosonic Lagrangian is given by

$$e^{-1} \mathcal{L} = R + \frac{2}{3}(A_{(1)}^2 - S^2 - P^2) + 4S \sqrt{-\Lambda/3} + \frac{1}{2} \alpha C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} - \frac{1}{3} \alpha F^{\mu\nu} F_{\mu\nu}, \quad (3)$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and $F = dA$. The supersymmetry transformation of the gravitino is given by

$$\delta\psi_\mu = -D_\mu\epsilon - \frac{i}{6}(2A_\mu - \Gamma_{\mu\nu}A^\nu)\Gamma_5\epsilon - \frac{1}{6}\Gamma_\mu(S + i\Gamma_5P)\epsilon. \quad (4)$$

The equations of motion for the scalar fields S and P imply that $S = 3\sqrt{-\Lambda/3}$ and $P = 0$. The vector equation of motion describes a massive Proca field: $\alpha\nabla^\mu F_{\mu\nu} + A_\nu = 0$. The Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - 2\alpha E_{\mu\nu} = -\frac{2}{3}(A_\mu A_\nu - \frac{1}{2}A^2 g_{\mu\nu}) + \frac{2}{3}\alpha(F_{\mu\nu}^2 - \frac{1}{4}F^2 g_{\mu\nu}),$$

where $E_{\mu\nu} = (\nabla^\rho\nabla^\sigma + \frac{1}{2}R^{\rho\sigma})C_{\mu\rho\sigma\nu}$ is the Bach tensor. It was shown in [8] that the theory admits a supersymmetric AdS vacuum with cosmological constant Λ , and the linearised spectrum of fluctuations around the AdS background was analyzed. The gravity modes are identical to those in Einstein-Weyl gravity, which was studied in [9]. There is a ghostlike massive spin-2 mode in addition to the massless graviton. The mass is determined by the product $\alpha\Lambda$. A special case, known as *Critical Gravity*, in which the massive mode is replaced by a mode with logarithmic fall-off, arises if $\alpha\Lambda = \frac{3}{2}$.

We consider a metric of the form (1) in $d = 4$, with the massive vector A_μ given by $A = qr^{-z}du$, where q is a constant. (The form of the r -dependence here is dictated by requiring invariance under the dilatation symmetry in (2).) We obtain solutions provided that

$$\alpha = -\frac{\ell^2}{z(z+1)}, \quad \Lambda = -\frac{3}{\ell^2}, \quad q = \pm\frac{3}{2}(z-1)\sqrt{c_1^2 + 2c_2}. \quad (5)$$

The solutions with $c_1 = 0$ were obtained in [10].

Supersymmetry analysis

Setting the AdS scale parameter $\ell = 1$ for convenience, we choose the vielbein basis

$$e^+ = du, \quad e^- = \frac{dv}{r^2} - \frac{c_1 dx}{r^{z+1}} + \frac{c_2 du}{2r^{2z}}, \quad e^2 = \frac{dx}{r}, \quad e^3 = \frac{dr}{r}, \quad (6)$$

such that $ds^2 = \eta_{ab}e^a \otimes e^b$ with $\eta_{+-} = -1$, $\eta_{22} = \eta_{33} = 1$. The inverse vielbein is given by

$$E_+ = \frac{\partial}{\partial u} - \frac{1}{2}c_2 r^{2-2z} \frac{\partial}{\partial v}, \quad E_- = r^2 \frac{\partial}{\partial v}, \quad E_2 = r \frac{\partial}{\partial x} + c_1 r^{2-z} \frac{\partial}{\partial v}, \quad E_3 = r \frac{\partial}{\partial r}. \quad (7)$$

The Lorentz-covariant exterior derivative $\nabla = d + \frac{1}{4}\omega^{ab}\Gamma_{ab}$ acting on spinors is then given, in terms of its vielbein components, by

$$\begin{aligned} \nabla_+ &= E_+ - \frac{1}{2}\Gamma_{+3} - \frac{1}{4}c_1(z-1)r^{-z}\Gamma_{23} - \frac{1}{2}c_2(z-1)r^{-2z}\Gamma_{-3}, & \nabla_- &= E_- - \frac{1}{2}\Gamma_{-3}, \\ \nabla_2 &= E_2 - \frac{1}{2}\Gamma_{23} + \frac{1}{4}c_1(z-1)r^{-z}\Gamma_{-3}, & \nabla_3 &= E_3 - \frac{1}{4}c_1(z-1)r^{-z}\Gamma_{-2}. \end{aligned} \quad (8)$$

Requiring that the supersymmetry variation of the gravitino, given by (4), vanish, we find that the solutions (5) are supersymmetric if any of the following holds:

- 1) $c_1 = 0, \quad q = 0: \quad \epsilon = r^{-1/2} \epsilon_-, \quad \text{where} \quad \Gamma_3 \epsilon_- = \epsilon_-, \quad \Gamma_- \epsilon_- = 0,$ (9)
- 2) $c_2 = 0, \quad q = \frac{3}{2} c_1 (z - 1): \quad \epsilon = r^{-1/2} \epsilon_+, \quad \text{where} \quad \Gamma_3 \epsilon_+ = \epsilon_+, \quad \Gamma_+ \epsilon_+ = 0,$
- 3) $c_2 = -\frac{4}{9} c_1^2, \quad q = -\frac{1}{2} c_1 (z - 1): \quad \epsilon = r^{-1/2} \epsilon_-, \quad \text{where} \quad \Gamma_3 \epsilon_- = \epsilon_-, \quad \Gamma_- \epsilon_- = 0.$

In each case the projection conditions imply that the spinor ϵ_{\pm} , which is constant, is unique up to scaling.

More general supersymmetric gyron solutions can also arise. As we have mentioned earlier, in general the constants c_i can be replaced by functions of all coordinates except v in gyrating geometries, and so one may consider metrics of the form

$$ds^2 = \frac{dr^2 - 2dudv + dx^2}{r^2} + f dudx + h du^2, \quad A = \phi du \quad (10)$$

where f , h and ϕ are functions of u , x and r . Here, we shall restrict our attention to the case where these functions depend only on r . The general such bosonic solution can easily be obtained explicitly. Since it is a little complicated to present, we shall not give it here. If in addition we require supersymmetry, then we find that the Killing spinor must satisfy the projections $\Gamma_3 \epsilon = \epsilon$ together with either $\Gamma_+ \epsilon = 0$ or $\Gamma_- \epsilon = 0$. In these two cases, the r -dependent functions must satisfy

$$\Gamma_+ \epsilon = 0: \quad \phi = -\frac{3}{2} r (f + \frac{1}{2} r f'), \quad h + \frac{1}{2} r h' = 0; \quad \Gamma_- \epsilon = 0: \quad \phi = \frac{1}{2} r (f + \frac{1}{2} r f'). \quad (11)$$

For generic values of z , related to α by $\alpha = -1/(z(z+1))$, imposing these supersymmetry restrictions on the general bosonic solution, and discarding trivial terms that can be immediately removed by coordinate transformations, we find:

$$\Gamma_+ \epsilon = 0: \quad f = \frac{a_1}{r^{1+z}} + a_2 r^z, \quad \phi = \frac{3(z-1)a_1}{4r^z} - \frac{3(z+2)a_2 r^{z+1}}{4}, \quad h = 0, \quad (12)$$

$$\Gamma_- \epsilon = 0: \quad f = \frac{a_1}{r^{1+z}} + a_2 r^z, \quad \phi = -\frac{(z-1)a_1}{4r^z} + \frac{(z+2)a_2 r^{z+1}}{4},$$

$$h = \frac{a_1^2}{9r^{2z}} + \frac{1}{9} a_2^2 r^{2z} + \frac{b_1}{r^{z+1}} + b_2 r^z + b_3 r. \quad (13)$$

A special case arises in the case of critical gravity, where $\alpha = -\frac{1}{2}$, i.e. where $z = 1$ or $z = -2$. We find the supersymmetric solutions with logarithmic fall-off

$$\Gamma_+ \epsilon = 0: \quad f = \frac{a_1 \log r}{r^2} + a_2 r, \quad \phi = -\frac{3a_1}{4r} - \frac{9}{4} a_2 r^2, \quad h = 0, \quad (14)$$

$$\Gamma_- \epsilon = 0: \quad f = \frac{a_1 \log r}{r^2} + a_2 r, \quad \phi = \frac{a_1}{4r} + \frac{3}{4} a_2 r^2,$$

$$h = \frac{a_1^2}{27r^2} (2 + 4 \log r + 3 \log^2 r) + \frac{1}{9} a_2^2 r^4 + \frac{b_1 \log r}{r^2} + b_2 r + b_3 r \log r. \quad (15)$$

3 Generalization to Higher Dimensions

Solutions in higher dimensions $d \geq 5$ can also arise. We find that Einstein gravity with additional quadratic curvature terms, coupled to a massive vector field, can support solutions with non-vanishing c_1 for general z . We consider the Lagrangian

$$e^{-1}\mathcal{L}_d = R - 2\Lambda + \alpha R^2 + \beta R^{\mu\nu}R_{\mu\nu} + \gamma E_{GB}^2 - \frac{1}{4}F^2 - \frac{1}{2}\sigma^2 A^2. \quad (16)$$

The equations of motion for the vector A_μ imply that $\sigma = z(z + d - 3)/\ell^2$. There are then three further equations from the variation of $g_{\mu\nu}$, and so with as-yet unspecified coefficients $(\alpha, \beta, \gamma, \sigma, \Lambda)$ in the theory, solutions must exist.

As an example, consider the case when $\alpha = \beta = 0$, corresponding to Gauss-Bonnet, or Lovelock, gravity. We then find

$$\Lambda = -\frac{(d-1)(d-2)}{4\ell^2}, \quad \gamma = \frac{\ell^2}{2(d-3)(d-4)}, \quad q^2 = -\frac{(z-1)^2 c_1^2 \ell^2}{4\ell^2}. \quad (17)$$

Note that c_2 is arbitrary in this case. The fact that q^2 is negative implies that we should really send $A_\mu \rightarrow i A_\mu$, implying that the massive vector is ghostlike.

As another example, we may consider Einstein-Weyl gravity in d dimensions, corresponding to taking

$$\alpha = \frac{4(d-3)}{d-2}\gamma, \quad \beta = -\frac{d(d-3)}{(d-1)(d-2)}. \quad (18)$$

We then find that

$$\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}, \quad \gamma = -\frac{(d-2)\ell^2}{4(d-3)z(z+d-3)},$$

$$q^2 = -\frac{\ell^2(z-1)^2}{z^2(z+d-3)} \left(2(d-4+3z)c_2 + \left(\frac{3}{4}z + \frac{(d-4)^2}{4(d-3)} \right) c_1^2 \right). \quad (19)$$

In the next section we shall require that $c_1^2 + c_2 > 0$. For suitable allowed choices of c_1 and c_2 , q^2 can be real in this case.

4 Boundary Field Theory

We consider a scalar field Φ with mass m_0 that is minimally coupled to the background metric (1). With

$$\Phi = f(r) e^{-i\omega t - ipv} e^{i\vec{k}\cdot\vec{y} + ik_x x}, \quad (20)$$

where $u = t$ is taken to be the time coordinate, the bulk wave equation $(\square - m_0^2)\Phi = 0$ gives

$$f'' - \frac{(n+2)}{r} f' + \left[-\frac{m_0^2}{r^2} - \frac{(c_1^2 + c_2)p^2}{r^{2z-2}} + \frac{2c_1 p k_x}{r^{z-1}} - \kappa^2 \right] f = 0, \quad (21)$$

where

$$\kappa^2 = \vec{k}^2 + k_x^2 + 2\omega p. \quad (22)$$

If one assumes that the coordinate v is compactified on a circle, then p will be quantised. Solutions of the form $f \sim r^\Delta$ near the boundary at $r = 0$ will exist if $z \leq 2$, and at the limiting value $z = 2$ equation (21) becomes

$$f'' - \frac{(n+2)}{r} f' + \left[-\frac{m^2}{r^2} + \frac{2c_1 p k_x}{r} - \kappa^2 \right] f = 0, \quad (23)$$

where

$$m^2 = m_0^2 + (c_1^2 + c_2) p^2. \quad (24)$$

It is necessary that $c_1^2 + c_2$ be non-negative in order that m^2 be positive for all values of the (quantised) v -momentum p . The asymptotic forms of the solutions at small r are $f \sim r^{\Delta_\pm}$ with

$$\Delta_\pm = \frac{1}{2}(n+3) \pm \nu, \quad \nu = \frac{1}{2}\sqrt{(n+3)^2 + 4m^2}. \quad (25)$$

The exact form of the general solution is

$$f = \alpha_1 r^{\Delta_+} M(a, b; 2\kappa r) e^{-\kappa r} + \alpha_2 r^{\Delta_-} U(a, b; 2\kappa r) e^{-\kappa r}, \quad (26)$$

where $M(a, b; 2\kappa r)$ and $U(a, b; 2\kappa r)$ are the confluent hypergeometric functions of the first and second kinds, and we have defined

$$a = \nu + \frac{1}{2} - \frac{c_1 p k_x}{\kappa}, \quad b = 2\nu + 1. \quad (27)$$

Demanding normalisability at large r requires that we reject the exponentially diverging solution involving $M(a, b; 2\kappa r)$, giving

$$f = r^{\Delta_+} U(a, b; 2\kappa r) e^{-\kappa r}, \quad (28)$$

which decays exponentially at infinity. At small r we use the relation

$$U(a, b; 2\kappa r) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b; 2\kappa r) + \frac{\Gamma(b-1)}{\Gamma(a)} (2\kappa r)^{1-b} M(a-b+1, 2-b; 2\kappa r), \quad (29)$$

which shows, since $M(a, b; 2\kappa r)$ is analytic at small r , that aside from contact terms the leading-order form for f is

$$f \sim (\kappa r)^{\Delta_-} \left[1 + \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(a-b+1)\Gamma(b-1)} r^{2\nu} \right]. \quad (30)$$

Thus, using the prescription [11]

$$G_R(k) = 2\mathcal{F}(k, r)|_{r=r_B} \sim \sqrt{-g} g^{rr} \partial_r \log f \quad (31)$$

for the retarded Green function, we find

$$\mathcal{F}(k, r)|_{r=\epsilon} \sim -\frac{(\epsilon\kappa)^{2\nu}}{\epsilon^{n+3}} \frac{\Gamma(1-2\nu)}{\Gamma(2\nu)} \frac{\Gamma(\nu + \frac{1}{2} - c_1 p k_x/\kappa)}{\Gamma(-\nu - \frac{1}{2} + c_1 p k_x/\kappa)}. \quad (32)$$

Unlike the situation in a pure Schrödinger spacetime background, where momentum dependence of the Green function enters only via the $(\epsilon\kappa)^{2\nu}$ factor, here when $c_1 \neq 0$ there is momentum dependence in the Gamma functions also.

5 Conclusions

In this paper, we have investigated certain four-dimensional generalisations of Schrödinger metrics that arise naturally as solutions of Einstein-Weyl gravity. Their symmetry group is smaller than that of the Schrödinger metrics, corresponding to an inisotropy not only between space and time, but also among the spatial coordinates themselves. The metrics are of Petrov type III, reducing to type N in the Schrödinger limit. Einstein-Weyl gravity can be viewed as the bosonic sector of an off-shell $\mathcal{N} = 1$ supergravity theory, and we find that included amongst the solutions we have obtained are some that are supersymmetric. We also considered higher-dimensional analogues of the four-dimensional solutions, and showed that they can also arise in Einstein-Weyl and in Einstein-Lovelock gravity in arbitrary dimensions $d > 4$.

The solutions we have obtained provide natural backgrounds for studying the dual strongly-coupled non-relativistic boundary theories. The spatial anisotropy would correspond to some breaking of rotational symmetry in the boundary theory, such as might arise from a uniform electric or magnetic field. We calculated the two-point correlation function for boundary operators dual to a minimally-coupled massive scalar probe field in the bulk theory.

Higher-order off-shell supergravities have rich structures for constructing geometries not only for relativistic but also for non-relativistic field theories, in the context of supersymmetry. It would be also of great interest to investigate whether there exist two-derivative gravity theories that also gives rise to gyrating Schrödinger geometries.

Acknowledgements

We are grateful to Sera Cremonini for useful discussions, and to the KITPC, Beijing, for hospitality during the course of this work. The research of H.L. is supported in part by NSFC grant 11175269. C.N.P. is supported in part by DOE grant DE-FG03-95ER40917.

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